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Hilbert Polynomials and Geometric Lattices

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1. INTRODUCTION

1.1. *The Setup of This Paper*

Let V be an l-dimensional vector space over a field $\mathbf{K}$. Fix a finite subset T of V. Then T determines a finite geometric lattice $L = L(T)$ [2, p. 80]: The elements of L are the closed subsets of T, which are in one-to-one correspondence to the vector subspaces spanned by subsets of T. (A subset X of T is called *closed* if $X = \langle X \rangle \cap T$, where $\langle X \rangle$ is the subspace spanned by X.) The partial order on L is defined by the inclusion $X \leqslant Y$ if $X \subseteq Y$. The minimum element is the empty set $\varnothing$ and the maximum element is T itself. Let S denote the symmetric algebra $S(V)$ of V. (Then S is isomorphic to the polynomial algebra over $\mathbf{K}$ in l variables.) Let $\mathscr{C}$ be the category of finitely generated graded S-modules with S-linear maps which are homogeneous of degree zero as morphisms. Regard the lattice L as a category with morphisms $\leqslant$

$$\operatorname{Hom}(X, Y) = \begin{cases} \{X \leqslant Y\} & \text{if } X \leqslant Y, \\ \varnothing & \text{otherwise.} \end{cases}$$

The composition of two morphisms $X \leqslant Y$ and $Y \leqslant Z$ is $X \leqslant Z$. Let F be a contravariant functor from L to $\mathscr{C}$. When $X, Y \in L$ with $Y \leqslant X$, the induced S-linear map homogeneous of degree zero from $F(X)$ to $F(Y)$ is denoted by $\nu_{X,Y}$ or simply by ν.

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1.2. *The Aim*

In this paper we will study the combinatorial properties of the Hilbert polynomials $H_{F(X)}(n)$ of the graded S-module $F(X)$, $X \in L$. Our formulae involve the coefficients of $H_{F(X)}(n)$ and the Möbius function of the lattice L.

1.3. *Local Functors*

It is of course nonsense to try to study the combinatorial properties of the Hilbert polynomial $H_{F(X)}(n)$ without any restriction, because the functor F may have nothing to do with the lattice structure of L, in other words, with the dependence relations among the vectors in T. So we restrict ourselves to the situation in which the functor F arises from T in "a geometric way." More precisely we only consider functors of the following type:

DEFINITION 1.3.1. A contravariant functor $F: L \to \mathscr{C}$ is said to be *local* if the localization of the S-linear map

$$\nu : F(X) \to F(X \cap \wp)$$

at $\wp$ is an isomorphism for any prime ideal $\wp$ of S:

$$\nu_{\wp} : F(X)_{\wp} \xrightarrow{\sim} F(X \cap \wp)_{\wp}.$$

(Note that $X \subseteq V \subset S$. So $X \cap \wp$ makes sense.)

EXAMPLE 1.3.2. Define a contravariant functor F from L to $\mathscr{C}$ by

$$F(X) = \left(\prod_{x \in X} x \right) S \qquad \text{(principal ideal)}.$$

The morphism ν is the inclusion map. Since x is a unit in the localized ring $S_{\wp}$ for $x \notin \wp$, F is local:

$$F(X)_{\wp} = \left(\prod_{x \in X} x \right) S_{\wp} = \left(\prod_{x \in X \cap \wp} x \right) S_{\wp} = F(X \cap \wp)_{\wp}.$$

1.4. *Hilbert Coefficients and a Theorem*

Let M be a finitely generated graded S-module. It is well known (e.g., see [5, Theorem 11, p. 320]) that the Hilbert polynomial of M can be uniquely written as

$$H_M(n) = h_0(M) \binom{n+l-1}{l-1} + h_1(M) \binom{n+l-2}{l-2} + \cdots + h_{l-1}(M) \binom{n}{0},$$

where the integers $h_0(M), h_1(M), \ldots, h_{l-1}(M)$ are called the *Hilbert coef-*

ficients of M. (We also define $h_l(M)$ by considering the cumulative Hilbert polynomial which is defined in 2.3.)

In Section 2, we will prove

THEOREM 2.8.1. *Let F be a local functor from L to $\mathscr{C}$ and r be the rank function on L. For $i < r(X)$,*

$$\sum_{\substack{Y \in L \\ Y \leqslant X}} \mu(Y, X)\, h_i(F(Y)) = 0.$$

Unless T contains the zero vector, for the local functor F in Example 1.3.2, Theorem 2.8.1 yields

$$\sum_{\substack{Y \in L \\ Y \leqslant X}} \mu(Y, X)(\# Y)^i = 0 \qquad (r(X) > i).$$

A large part of the proof of Theorem 2.8.1 is embedded in [8], where Hilbert polynomials or Hilbert coefficients did not appear on the surface. The corresponding result in [8] in Proposition (6.10). It was used to prove a formula for the characteristic polynomial of an arrangement of hyperplanes.

In [4, p. 311], Kelly and Rota suggested the relation between the Hilbert coefficients and the Möbius functions. Theorem 2.8.1 gives a relation of that type.

1.5. *The Functors D^p*

In Section 3 we study an arrangement $\mathscr{A}$ of hyperplanes in $\mathbf{K}^l$: $\mathscr{A}$ is a finite family of hyperplanes (=subspaces of codimension one) of $\mathbf{K}^l$. Let V be the dual vector space of $\mathbf{K}^l$. For each $H \in \mathscr{A}$, choose $\alpha_H \in V$ such that $\ker(\alpha_H) = H$. Let

$$T = \{\alpha_H \in V \mid H \in \mathscr{A}\}.$$

Define L, S, and $\mathscr{C}$ as in 1.1. Then L is isomorphic to the intersection lattice $L(\mathscr{A})$ [8, p. 305] of the arrangement $\mathscr{A}$. The $\mathbf{K}$-algebra $S = S(V) = S((\mathbf{K}^l)^*)$ is identified with the $\mathbf{K}$-algebra of polynomial functions on $\mathbf{K}^l$. In Section 3 we study special contravariant functors D^p $(0 \leqslant p \leqslant l)$ from L to $\mathscr{C}$, which were studied in [8]. First, in Proposition 3.2.2, we will show that each D^p is a local functor. Next we will prove that the first three Hilbert coefficients $h_i(D^p(X))$ $(i = 0, 1, 2)$ are combinatorial invariants in 3.4.3. In other words, they are determined by (X and) the lattice L. We actually have the explicit combinatorial expressions in 3.4.2 for these combinatorial invariants. We also give an example $(l = 3)$ in 3.5.1 which shows that the last Hilbert coefficient, $h_3(D^p(T))$, is not necessarily a combinatorial invariant. Although each $h_3(D^p(T))$ $(0 \leqslant p \leqslant l)$ is not a combinatorial invariant in this example, the alternating sum of these is a combinatorial invariant. More generally, in 3.6.3 we will show

$$\mu(\varnothing, T) = (-1)^l \sum_{p=0}^{l} (-1)^p h_l(D^p(T)),$$

when the arrangement $\mathscr{A}$ is essential (or equivalently T spans V). This is obtained from the Main Theorem (1.2) in [8] and results from Section 2.

Let $\Omega^p(X)$ be the module of logarithmic p-forms, which was first studied in [7]. All the results concerning the functors D^p in Section 3 will be obtained for the functors Ω^p $(0 \leqslant p \leqslant l)$ in 3.7.

Part of this work was done while the second author spent the academic year 1987–88 at the Ohio State University. He expresses his gratitude to their hospitality.

2. Local Functors and the Hilbert Coefficients

2.1. *The Setup*

Let $\mathbf{K}$, V, l, T, S, L, $\mathscr{C}$ be as in 1.1. In this section we assume that F is a *local* contravariant functor from L to $\mathscr{C}$. (The theoretical part of this paper is still true when the functor F is covariant if it is modified in the obvious way. We choose to use *contravariant* functors because in our important examples F is contravariant.)

2.2. *Intersection with a Prime Ideal*

Recall that the expression $F(X \cap \wp)$ appeared in Definition 1.3.1 of a local functor. For the definition to be well defined, we need the following:

Proposition 2.2.1. *Let $X \in L$ and let $\wp$ be an ideal of S. Then $X \cap \wp \in L$. (In this case, obviously $X \cap \wp \leqslant X$.)*

Proof. Suppose $x \in T$ is dependent over $X \cap \wp$. Then x is in X because X is closed. Since x is a linear combination of elements of $\wp$, it belongs to $\wp$. ∎

2.3. *The Hilbert Polynomial*

Let $M = \oplus_{n \geqslant 0} M_n$ be a finitely generated graded S-module. Recall that the Hilbert polynomial $H_M(n) \in \mathbf{Q}[n]$ of M is characterized by

$$H_M(n) = \dim_{\mathbf{K}} M_n \qquad (n \gg 0).$$

Recall that the degree of the polynomial $H_M(n)$ is equal to $\dim M - 1$ ($\dim M$ is the dimension of M as an S-module) [5, Theorem 19, p. 346]. Since $\dim M \leqslant \dim S \leqslant l$, one knows that $\deg H_M \leqslant l - 1$.

Sometimes it is convenient to consider the *cumulative* Hilbert polynomial $H_M^*(n) \in \mathbf{Q}[n]$ characterized by

$$H_M^*(n) = \sum_{i=0}^{n} \dim_{\mathbf{K}} M_i \qquad (n \gg 0)$$

(cf. [5, p. 317]). Then $\deg H_M^* = 1 + \deg H_M = \deg M \leqslant l$.

2.4. *The Hilbert Coefficients*

Since the cumulative Hilbert polynomial $H_M^*(n) \in \mathbf{Q}[n]$ is a polynomial of degree at most l such that $H_M^*(n) \in \mathbf{Z}$ for all $n \gg 0$, $n \in \mathbf{Z}$, it is well known that there are uniquely determined integers $h_i(M)$ ($i = 0, 1, ..., l$) such that

$$H_M^*(n) = h_0(M)\binom{n+l}{l} + h_1(M)\binom{n+l-1}{l-1} + \cdots + h_l(M)\binom{n}{0}.$$

DEFINITION 2.4.1. The integers $h_i(M)$ $(0 \leqslant i \leqslant l)$ are called the *Hilbert coefficients* of M. (The first one $h_0(M)$, which might be zero, is equal to the *rank* of M.)

Remark. It is easy to see that the Hilbert polynomial $H_M(n)$ is written as

$$H_M^*(n) - H_M^*(n-1) = h_0(M)\binom{n+l-1}{l-1} + h_1(M)\binom{n+l-2}{l-2} + \cdots + h_{l-1}(M)\binom{n}{0}.$$

Usually these l integers $h_0(M), ..., h_{l-1}(M)$ are called the Hilbert coefficients as in [5, p. 323]. In this paper we also consider $h_l(M)$.

2.5. *The Poincaré Series*

The *Poincaré series* $\mathrm{Poin}(M; x)$ (also called *Hilbert series*) of M is a formal power series in x defined by

$$\mathrm{Poin}(M; x) = \sum_{n \geqslant 0} (\dim_{\mathbf{K}} M_n)\, x^n.$$

Define

$$P(M; x) = (1-x)^l\, \mathrm{Poin}(M; x).$$

LEMMA 2.5.1. *$P(M; x)$ is a polynomial in x such that*

$$h_i(M) = (-1)^i P^{(i)}(M; 1)/i! \qquad (0 \leqslant i \leqslant l),$$

where $P^{(i)}(M; 1)$ is the value at $x = 1$ of the ith derivative of $P(M; x)$.

Proof. *Case* 1 ($0 \leqslant i < l$). We have

$$\begin{aligned}\sum_{n \geqslant 0} H_M(n)\, x^n &= \sum_{n \geqslant 0} \sum_{k=0}^{l-1} h_k(M) \binom{n+l-1-k}{l-1-k} x^n \\ &= \sum_{k=0}^{l-1} h_k(M) \sum_{n \geqslant 0} \binom{n+l-1-k}{l-1-k} x^n \\ &= \sum_{k=0}^{l-1} h_k(M)(1-x)^{k-l}.\end{aligned}$$

Now the difference

$$A(x) := \mathrm{Poin}(M; x) - \sum_{n \geqslant 0} H_M(n)\, x^n \tag{1}$$

is a polynomial. Multiply $(1-x)^l$ to both sides to get

$$(1-x)^l A(x) = P(M; x) - \sum_{k=0}^{l-1} h_k(M)(1-x)^k. \tag{2}$$

Differentiating both sides i times and putting $x = 1$, one gets the result.

Case 2 ($i = l$). In (1) compare the sum of coefficients of x^k ($0 \leqslant k \leqslant n$) for $n \gg 0$. Then we get

$$\begin{aligned}A(1) &= H^*_M(n) - \sum_{k=0}^{n} H_M(k) \\ &= H^*_M(n) - H_M(0) - \sum_{k=1}^{n} (H^*_M(k) - H^*_M(k-1)) \\ &= H^*_M(0) - H_M(0) = h_l(M).\end{aligned}$$

Differentiating both sides of (2) l times and evaluating when $x = 1$ gives

$$(-1)^l (l!)\, A(1) = P^{(l)}(M; 1).$$

So

$$h_l(M) = A(1) = (-1)^l P^{(l)}(M; 1)/(l!). \quad \blacksquare$$

2.6. *Notation*

Let X and Z be in L. Define

$$L_X = \{Y \in L \mid Y \leqslant X\} \qquad \text{(lower sublattice)}$$

and

$$L_X^Z = \{Y \in L \mid Z \leqslant Y \leqslant X\} \qquad \text{(interval)}.$$

For a nonnegative integer i, define

$$L_X(i) = \{Y \in L \mid Y \leqslant X, r(Y) = i\}$$

and

$$L_X^Z(i) = \{Y \in L \mid Z \leqslant Y \leqslant X, r(Y) = i\}.$$

2.7. *A Result of Solomon–Terao*

PROPOSITION 2.7.1 [8, 6.7, 6.10]. *Assume that F is a local functor from L to $\mathscr{C}$. For any $X \in L$,*

$$\sum_{Y \in L_X} \mu(Y, X) \operatorname{Poin}(F(Y); x)$$

has a pole of order at most $l - r(X)$ at $x = 1$. Here μ is the Möbius function on L [6] and r is the rank function on L.

Remark. In [8] the situation was actually slightly different. In order to prove Proposition 2.7.1 above, we have to replace codim by r, $X \wedge T(\wp)$ by $X \cap \wp$, and dim X by $l - r(X)$ in (6.8)–(6.12) in [8].

2.8. *Two Theorems*

Because of Lemma 2.5.1, Proposition 2.7.1 is reformulated as

THEOREM 2.8.1. *Assume that F is a local functor from L to $\mathscr{C}$. For any $X \in L$ and $0 \leqslant i < r(X)$,*

$$\sum_{Y \in L_X} \mu(Y, X)\, h_i(F(Y)) = 0.$$

EXAMPLE 2.8.2. Let

$$F(X) = \left(\prod_{x \in X} x\right) S$$

as in Example 1.3.2. Suppose that T does not contain the zero vector (Otherwise $F(X)=0$ for all $X \in L$.) Then

$$\operatorname{Poin}(F(X); x) = x^{\# X}/(1-x)^l$$

and

$$P(F(X); x) = x^{\# X}.$$

So by Lemma 2.5.1

$$h_i(F(X)) = (-1)^i \binom{\# X}{i} \qquad (0 \leqslant i \leqslant l).$$

Then Theorem 2.8.1 yields

$$\sum_{Y \in L_X} \mu(Y, X) \binom{\# Y}{i} = 0 \qquad (r(X) > i).$$

From these we inductively obtain

$$\sum_{Y \in L_X} \mu(Y, X)(\# Y)^i = 0 \qquad (r(X) > i).$$

THEOREM 2.8.3. *For any* $X \in L$ *and* $0 \leqslant i \leqslant l$,

$$h_i(F(X)) = \sum_{\substack{Z \in L_X \\ r(Z) \leqslant i}} h_i(F(Z)) \sum_{\substack{Y \in L_X^Z \\ r(Y) \leqslant i}} \mu(Z, Y).$$

Proof. Define

$$f(X) = h_i(F(X)) \qquad (X \in L)$$

and

$$g(X) = \begin{cases} \sum_{Y \in L_X} \mu(Y, X)\, h_i(F(Y)) & \text{if } \ r(X) \leqslant i \\ 0 & \text{if } \ r(X) > i \end{cases}$$

for $X \in L$. Then by Theorem 2.8.1

$$\sum_{Y \in L_X} \mu(Y, X)\, f(Y) = g(X)$$

for $X \in L$. Using the Möbius inversion formula (e.g., [9, p. 116]) we have

$$\begin{aligned} f(X) &= \sum_{Y \in L_X} g(Y) \\ &= \sum_{\substack{Y \in L_X \\ r(Y) \leqslant i}} \sum_{Z \in L_Y} \mu(Z, Y)\, h_i(F(Z)) \\ &= \sum_{\substack{Z \in L_X \\ r(Z) \leqslant i}} h_i(F(Z)) \sum_{\substack{Y \in L_X^Z \\ r(Y) \leqslant i}} \mu(Z, Y). \quad \blacksquare \end{aligned}$$

2.9. *When i Is Small*

When F is a local functor and $i = 0, 1, 2$, Theorem 2.8.3 implies the following results:

$(i=0)$ $h_0(F(X)) = h_0(F(\varnothing))$ for all $X \in L$. (In other words the rank of the S-module $F(X)$ is independent of $X \in L$.)

$$(i=1) \quad h_1(F(X)) = h_1(F(\varnothing)) \Bigg(\sum_{\substack{Y \in L_X \\ r(Y) \leqslant 1}} \mu(\varnothing, Y) \Bigg) + \sum_{Z \in L_X(1)} h_1(F(Z))$$

$$= h_1(F(\varnothing))(1 - \# L_X(1)) + \sum_{Z \in L_X(1)} h_1(F(Z)).$$

$$(i=2) \quad h_2(F(X)) = h_2(F(\varnothing)) \Bigg(1 - \# L_X(1) + \sum_{Y \in L_X(2)} \mu(\varnothing, Y) \Bigg)$$

$$- \sum_{Z \in L_X(1)} h_2(F(Z))(\# L_X^Z(2)) + \sum_{Z \in L_X(2)} h_2(F(Z)).$$

EXAMPLE 2.9.1. Suppose that any two vectors of T are linearly independent. (Then $L_X(1) = X$ for all $X \in L$.) Let

$$F(X) = \Bigg(\prod_{x \in X} x \Bigg) S.$$

Then as we saw in Example 2.8.2,

$$h_i(F(X)) = (-1)^i \binom{\# X}{i} \qquad (0 \leqslant i \leqslant l).$$

Note that $h_1(F(\varnothing)) = 0 = h_2(F(Z))$ if $r(Z) = 1$. Thus Theorem 2.8.3 in this case for $i = 2$ yields

$$(\# X)(\# X - 1) = \sum_{Z \in L_X(2)} (\# Z)(\# Z - 1).$$

This is a well-known (but non-trivial) formula. (For a finite family of lines in $\mathbf{R}^2$ this corresponds to the famous formula (e.g., [3, p. 16])

$$\binom{n}{2} = \sum_{j \geqslant 2} \binom{j}{2} t_j,$$

where n is the number of lines and t_j is the number of points through which there are precisely j lines.)

3. The Functors D^p

3.1. *The Setup*

Let $\mathscr{A}$, $\mathbf{K}$, l, V, T, L, S be as in 1.5. Then L is isomorpic to the intersection lattice $L(\mathscr{A})$, which is the collection of all intersections of elements of $\mathscr{A}$ partially ordered by the reverse inclusion. The rank function $r: L \to \mathbf{Z}$ satisfies

$$r(X) = \dim \langle X \rangle = \operatorname{codim} Z(X),$$

where $\langle X \rangle$ is the subspace of V by X and $Z(X) \subseteq \mathbf{K}^l$ is the common zero set of X. In this section we study special local contravariant functors D^p $(0 \leqslant p \leqslant l)$ from L to $\mathscr{C}$. The module $D^1(X)$, which is sometimes denoted simply by $D(X)$, is identified with the module of logarithmic derivations (or vector fields) studied in [7, 10] among others. The functors D^p $(2 \leqslant p \leqslant l)$ were introduced in [8] and were used to give a formula (Theorem 3.6.1 in this section) for the characteristic polynomial of an arrangement of hyperplanes.

3.2. *The Functors D^p*

Definition 3.2.1. For $X \in L$ and $0 \leqslant p \leqslant l$, let

$$D^p(X) = \left\{ \theta \in \operatorname{Hom}_{\mathbf{K}}\left(\bigwedge^p V, S \right) \,\middle|\, \theta\left(x \wedge \bigwedge^{p-1} V \right) \in xS \text{ for all } x \in X \right\}.$$

An element $\theta \in D^p(X)$ is called *homogeneous* of degree d if $\operatorname{im} \theta \subseteq S_d$ (S_d is the homogeneous part of S of degree d.) It is easy to see that $D^p(X)$ is a finitely generated graded S-module in the obvious way. When $X, Y \in L$, $Y \leqslant X$, we have $D^p(X) \subseteq D^p(Y)$. So D^p can be considered as a contravariant functor from L to $\mathscr{C}$ with the inclusion map $D^p(X) \hookrightarrow D^p(Y)$ as $\nu = \nu_{X,Y}$.

Remark. Each element $\theta: \bigwedge^p V \to S$ can be uniquely extended to a $\mathbf{K}$-linear map $\bar{\theta}: \bigwedge^p S \to S$ such that $\bar{\theta}$ is a derivation in each variable:

$$\bar{\theta}(fg \wedge g_2 \wedge \cdots \wedge g_p) = f\bar{\theta}(g \wedge g_2 \wedge \cdots \wedge g_p) + g\bar{\theta}(f \wedge g_2 \wedge \cdots \wedge g_p)$$

for $f, g, g_2, ..., g_p \in S$. In [8] the map $\bar{\theta}$, instead of θ, was studied. Also note that the module $D^p(\mathscr{A})$ in [8] is the same as $D^p(T)$ here.

PROPOSITION 3.2.2. *Each D^p $(0 \leqslant p \leqslant l)$ is local.*

Proof. Since $D^p(X) \subseteq D^p(X \cap \wp)$, $D^p(X)_\wp \subseteq D^p(X \cap \wp)_\wp$. Let

$$\theta/f \in D^p(X \cap \wp)_\wp,$$

for $\theta \in D^p(X \cap \wp)$, $f \in S \backslash \wp$. Define $g = \prod_{x \in X \backslash \wp} x$. Then

$$\theta/f = g\theta/gf \in D^p(X)_\wp.$$

So

$$D^p(X)_\wp \supseteq D^p(X \cap \wp)_\wp. \quad \blacksquare$$

3.3. *When $D(X)$ Is Free*

From now on simply denote D^1 by D. When $D(X)$ is a free S-module, there exists a homogeneous basis $\theta_1, ..., \theta_l$ for $D(X)$. Let

$$d_i = \deg \theta_i \qquad (1 \leqslant i \leqslant l).$$

The nonnegative integers $d_1, ..., d_l$ are called the *exponents* (of $D(X)$).

Remark. In particular when $D(T)$ is free, we call the arrangement $\mathscr{A}$ a *free arrangement*. The *exponents of $\mathscr{A}$* are the exponents of $D(T)$ here.

The following proposition [8, 3.7] is not difficult to check:

PROPOSITION 3.3.1. *If $D(X)$ is a free module with exponents $d_1, ..., d_l$, then*

$$\sum_{p=0}^{l} \operatorname{Poin}(D^p(X); x)\, y^p = (1-x)^{-l} \prod_{i=1}^{l} (1 + x^{d_i} y).$$

Equivalently

$$\sum_{p=0}^{l} P(D^p(X); x)\, y^p = \prod_{i=1}^{l} (1 + x^{d_i} y).$$

Here y is an indeterminate.

3.4. *The First Three Hilbert Coefficients Are Combinatorial Invariants*

Let $x_1, ..., x_l$ be a basis for V and $\partial/\partial x_i \in \mathrm{Hom}(V, S)$ such that $(\partial/\partial x_i)(x_j) = \delta_{ij}$ (Kronecker's delta) $(1 \leqslant i, j \leqslant l)$. Then the special element

$$\theta_E = \sum_{i=1}^{l} x_i(\partial/\partial x_i)$$

lies in $D(X)$ for all $X \in L$. θ_E is called the *Euler derivation.*

LEMMA 3.4.1. *Let $Z \in L$ and r be the rank function. Let $0 \leqslant p \leqslant l$. Suppose $l \geqslant 3$. Then*

(1) *If $r(Z) = 0$ (i.e., $Z = \varnothing$), then $h_0(D^p(Z)) = \binom{l}{p}$ and $h_i(D^p(Z)) = 0$ $(1 \leqslant i \leqslant l)$;*

(2) *if $r(Z) = 1$ (i.e., $\# Z = 1$), then $h_1(D^p(Z)) = -\binom{l-1}{p-1}$, and $h_i(D^p(Z)) = 0$ $(2 \leqslant i \leqslant l)$;*

(3) *if $r(Z) = 2$, then*

$$h_2(D^p(Z)) = \binom{l-2}{p-2}\binom{m}{2} + \binom{l-2}{p-1}\binom{m-1}{2},$$

where $m = \# Z$.

Proof. (1) When $Z = \varnothing$, $D(Z) = \mathrm{Hom}(V, S)$ is a free S-module with basis $\partial/\partial x_1, ..., \partial/\partial x_l$. So the exponents are all 0. By Proposition 3.3.1, we have

$$P(D^p(Z); x) = \binom{l}{p}.$$

Apply Lemma 2.5.1.

(2) When $\# Z = 1$, we can choose a basis $x_1, x_2, ..., x_l$ for V such that $Z = \{x_1\}$. Then $D(Z)$ is a free S-module with basis $x_1(\partial/\partial x_1)$, $\partial/\partial x_2, ..., \partial/\partial x_l$. So the exponents are $1, 0, 0, ..., 0$. By Proposition 3.3.1, we have

$$P(D^p(Z); x) = \binom{l-1}{p} + \binom{l-1}{p-1} x.$$

Apply Lemma 2.5.1.

(3) When $r(Z) = 2$, we can assume that every element of Z is a linear combination of x_1 and x_2 and that $Z = \{x_1, \alpha_2, \alpha_3, ..., \alpha_m\}$. It is not difficult to see that $D(Z)$ is a free S-module with a basis consisting of θ_E (Euler

derivation), $(\alpha_2 \cdots \alpha_m)(\partial/\partial x_2)$, $\partial/\partial x_3, ..., \partial/\partial x_l$. So the exponents are $m-1, 1, 0, 0, ..., 0$. By Proposition 3.3.1, we have

$$P(D^p(Z); x) = \binom{l-2}{p} + \binom{l-2}{p-1}(x + x^{m-1}) + \binom{l-2}{p-2} x^m.$$

Apply Lemma 2.5.1. ▮

THEOREM 3.4.2. *Let $X \in L$ and $0 \leqslant p \leqslant l$. Then*

(1) $h_0(D^p(X)) = \binom{l}{p}$,

(2) $h_1(D^p(X)) = -\binom{l-1}{p-1}(\#X)$,

(3) $h_2(D^p(X)) = \sum_{Z \in L_X(2)} \{\binom{l-2}{p-2}\binom{\#Z}{2} + \binom{l-2}{p-1}\binom{\#Z-1}{2}\}$.

Proof. Easy from the formulae in 2.9 and Lemma 3.4.1. ▮

Remark. When $p=1$, (2) of the preceding theorem was proved in [1, Theorem 4.6(b)], using methods from commutative and homological algebra. The methods also work for all p, $0 \leqslant p \leqslant l$.

COROLLARY 3.4.3. *The first three Hilbert coefficients $h_i(D^p(X))$ $(i = 0, 1, 2)$ are combinatorial invariants.*

3.5. $h_3(D^p(X))$ *Is Not a Combinatorial Invariant*

EXAMPLE 3.5.1. The following arrangements were defined by G. Ziegler [11, Example 4.1].

Let $\mathbf{K}$ be a field containing ω with $\omega^2 - \omega + 1 = 0$. Consider an arrangement in $\mathbf{K}^3$ consisting of the following 9 planes:

$$x=0, \quad y=0, \quad z=0, \quad y+z=0, \quad x+y+z=0,$$

$$x+\omega y=0, \quad \omega x+z=0, \quad \omega x+\omega y+z=0, \quad x+\omega y+z=0.$$

Then

$$T = \{x, y, z, y+z, x+y+z, x+\omega y, \omega x+z, \omega x+\omega y+z, x+\omega y+z\} \subseteq V = (\mathbf{K}^3)^*.$$

Case 1. When $\mathbf{K} = \mathbf{Z}_{13}$ and $\omega = 4$, Ziegler showed that $D(T)$ is a free module with exponents 1, 4, 4. By Proposition 3.3.1 we have

$$P(D^1(T); x) = x + 2x^4, \qquad P(D^2(T); x) = x^8 + 2x^5, \qquad P(D^3(T); x) = x^9.$$

Apply Lemma 2.5.1, and we have

$$h_3(D^1(T)) = -8, \qquad h_3(D^2(T)) = -76, \qquad h_3(D^3(T)) = -84.$$

Case 2. When $\mathbf{K}=\mathbf{Z}_3$ and $\omega=2$, Ziegler showed that $D(T)$ is not a free module. By using a computer algebra system MACAULAY we computed

$$P(D^1(T);x)=x+x^3+2x^6-x^7, \qquad P(D^2(T);x)=x^4+2x^7,$$
$$P(D^3(T);x)=x^9.$$

Apply Lemma 2.5.1, and we have

$$h_3(D^1(T))=4, \qquad h_3(D^2(T))=-64, \qquad h_3(D^3(T))=-84.$$

Ziegler [11, Example 4.1] showed that these two arrangements share the same lattice L. Thus the Hilbert coefficient $h_3(D^p(T))$ is not a combinatorial invariant in general.

Note that the alternating sums are equal:

$$(-8)-(-76)+(-84)=-16=4-(-64)+(-84).$$

The number -16 is equal to the value $\mu(\emptyset, T)$ of the Möbius function. This is proved in general in 3.6.3.

3.6. *A Formula for* $\mu(\emptyset, T)$

In [8, 5.3, 1.2] the following theorem was proved:

THEOREM 3.6.1. (i) $\Psi(x,t):=\sum_{p=0}^{l}\operatorname{Poin}(D^p(X);x)\{t(x-1)-1\}^p$ *is a polynomial in* x *and* t.

(ii) $\sum_{X\in L}\mu(\emptyset,X)\,t^{l-r(X)}=(-1)^l\,\Psi(1,t)$.

THEOREM 3.6.2. *Let* $r=r(T)$. (*In other words, r is the dimension of the subspace spanned by T.*) *Then*

$$(-1)^l\sum_{p=0}^{l}(-1)^p\binom{p}{j}h_i(D^p(T))=\begin{cases}0 & \text{if } i+j<l\\ \mu(\emptyset,T) & \text{if } i=r,\ j=l-r.\end{cases}$$

Proof. By Lemma 2.5.1, there is a polynomial $B(x)$ such that

$$P(D^p(X);x)=\sum_{i=0}^{l}h_i(D^p(X))(1-x)^i+(1-x)^{l+1}B(x).$$

Therefore

$$\begin{aligned}\Psi(x,t)&=\sum_{p=0}^{l}\operatorname{Poin}(D^p(X);x)\{t(x-1)-1\}^p\\ &=\sum_{p=0}^{l}\left\{\sum_{i=0}^{l}h_i(D^p(X))(1-x)^{i-l}+(1-x)\,B(x)\right\}\{t(x-1)-1\}^p.\end{aligned}$$

By Theorem 3.6.1(i), the coefficient of t^j should be a polynomial in x. Thus

$$\sum_{p=0}^{l} (-1)^p \binom{p}{j} h_i(D^p(T)) = 0 \qquad (i+j<l).$$

Next the coefficient of t^{l-r} in $\Psi(1, t)$ is

$$\sum_{p=0}^{l} h_r(D^p(X))(-1)^p \binom{p}{l-r}.$$

Now, since T is the maximum element of L, we have

$$\sum_{X \in L} \mu(\varnothing, X)\, t^{l-r(X)} = \mu(\varnothing, T)\, t^{l-r} + \text{(higher terms)}.$$

Therefore, by Theorem 3.6.1(ii), we obtain the desired result. ∎

COROLLARY 3.6.3. *When the arrangement* $\mathscr{A}$ *is essential* (*i.e., the intersection of all the hyperplanes in* $\mathscr{A}$ *is just the origin*),

$$\mu(\varnothing, T) = (-1)^l \sum_{p=0}^{l} (-1)^p h_l(D^p(T)).$$

Proof. Note that $\mathscr{A}$ is essential if and only if T spans V. So $r(T) = l$. Apply Theorem 3.6.2. ∎

3.7. *Remarks about the Functors* Ω^p

For $X \in L$, let

$$Q(X) = \prod_{x \in X} x.$$

Define

$$\Omega^p(X) = \{\omega \mid \omega \text{ is a rational } p\text{-form on } \mathbf{K}^l \text{ such that both } Q(X)\,\omega \text{ and } Q(X)(d\omega) \text{ have no poles}\}.$$

Then $\Omega^p(X)$ is naturally an S-module. The grading is introduced so that

$$\deg(dx) = 0, \qquad \deg(x\, dx) = 1 \qquad (x \in V).$$

Let

$$m = \# X = \deg Q(X).$$

Then

$$\deg(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_l/Q(X)) = -m$$

and

$$\Omega^p(X) = \bigoplus_{k \geqslant -m} \Omega^p(X)_k.$$

The graded S-module $\Omega^p(X)$ is called the module of *logarithmic differential p-forms.*

It is not difficult to see that there is an isomorphism

$$\phi: \Omega^p(X)(-m) \simeq D^{l-p}(X)$$

as graded modules. Here $(-m)$ stands for the shift of grading by $-m$. Explicitly ϕ is given by

$$\phi\left(\sum_I f_I\, dx_I/Q(X)\right) = \sum_I f_I\, D_{I^c},$$

where $I = (i_1, i_2, ..., i_p)$, $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, $I^c = \{1, 2, ..., l\} \backslash I = (j_1, j_2, ..., j_{l-p})$, $D_{I^c} = \partial/\partial x_{j_1} \wedge \cdots \wedge \partial/\partial x_{j_{l-p}}$. Therefore we have

$$x^m \operatorname{Poin}(\Omega^p(X); x) = \operatorname{Poin}(D^{l-p}(X); x),$$

$$H^*_{\Omega^p(X)}(n) = H^*_{D^{l-p}(X)}(n+m).$$

Hence we easily have

$$h_i(\Omega^p(X)) = \sum_{j=0}^{i} \binom{m+j-1}{j} h_{i-j}(D^{l-p}(X)) \qquad (0 \leqslant i \leqslant l).$$

So it is an easy task to get the results of this section for $\Omega^p(X)$. We just list them:

$$h_0(\Omega^p(X)) = \binom{l}{p},$$

$$h_1(\Omega^p(X)) = \binom{l-1}{p-1}(\# X),$$

$$h_2(\Omega^p(X)) = \binom{l}{p}\binom{1+\# X}{2} - \binom{l-1}{p}(\# X)^2$$

$$+ \sum_{Z \in L_X(2)} \left\{\binom{l-2}{p}\binom{\# Z}{2} + \binom{l-2}{p-1}\binom{\# Z - 1}{2}\right\},$$

$$\sum_{p=0}^{l} (-1)^p \binom{l-p}{j} h_i(\Omega^p(T)) = \begin{cases} 0 & \text{if } \; i+j<l \\ \mu(\varnothing, T) & \text{if } \; i=r,\ j=l-r. \end{cases}$$

Here $r = r(T)$.

For Ziegler's example 3.5.1, we have

$$h_3(\Omega^1(T)) = \begin{cases} 41 & \text{(Case 1)} \\ 43 & \text{(Case 2)}, \end{cases} \qquad h_3(\Omega^2(T)) = \begin{cases} 190 & \text{(Case 1)} \\ 192 & \text{(Case 2)}, \end{cases}$$

$$h_3(\Omega^3(T)) = 165 \quad \text{(Cases 1, 2)},$$

$$-41 + 190 - 165 = -16 = \mu(\varnothing, T) = -43 + 192 - 165.$$

So $h_3(\Omega^p(X))$ is not, in general, a combinatorial invariant either.

References

1. L. J. Billera and L. L. Rose, A dimension series for multivariate splines, *Discrete Comput. Geom.*, in press.
2. G. Birkhoff, "Lattice Theory," 3rd ed., Amer. Math. Soc. Colloquium Publ., Vol. 25, Amer. Math. Soc., Providence, RI, 1967.
3. B. Grünbaum, Arrangements and spreads, *in* "Regional Conference Series in Math.," Vol. 10, Amer. Math. Soc., Providence, RI, 1971.
4. D. Kelly and G.-C. Rota, Some problems in combinatorial geometry, *in* "A Survey of Combinatorial Theory" (J. N. Srivastava *et al.*, Eds.), pp. 309–312, North-Holland, Amsterdam, 1973.
5. D. G. Northcott, "Lessons on Rings, Modules and Multiplicities," Cambridge Univ. Press, London/New York, 1968.
6. G.-C. Rota, On the foundations of combinatorial theory. I. Theory of Möbius functions, *Z. Wahrsch.* **2** (1964), 340–368.
7. K. Saito, Theory of logarithmic differential forms and logarithmic vector fields, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 265–291.
8. L. Solomon and H. Terao, A formula for the characteristic polynomial of an arrangement, *Adv. in Math.* **64** (1987), 305–325.
9. R. P. Stanley, "Enumerative Combinatorics," Vol. I, Wadsworth & Brooks/Cole, Monterey, CA, 1986.
10. H. Terao, Generalized exponents of a free arrangement of hyperplanes and Shepherd–Todd–Brieskorn formula, *Invent. Math.* **63**, No. 1 (1981), 159–179.
11. G. Ziegler, Matroid representations and free arrangements, *in* "Schwerpunktprogramm der Deutschen Forschungsgemeinschaft, Anwendungsbezogene Optimierung und Steuerung," Vol. 77, 1988.